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Role of saddles in mean-field dynamics above the glass transition

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Abstract

Recent numerical developments in the study of glassy systems have shown that it is possible to give a purely geometric interpretation of the dynamic glass transition by considering the properties of unstable saddle points of the energy. Here we further develop this approach in the context of a mean-field model, by analytically studying the properties of the closest saddle point to an equilibrium configuration of the system. We prove that when the glass transition is approached the energy of the closest saddle goes to the threshold energy, defined as the energy level below which the degree of instability of the typical stationary points vanishes. Moreover, we show that the distance between a typical equilibrium configuration and the closest saddle is always very small and that, surprisingly, it is almost independent of the temperature.

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The glass transition occurs when the relaxation time of a substance increases upon cooling of many orders of magnitude in a very narrow interval of temperature, without the onset of any crystalline order. Even though dramatic changes in the mechanical properties of the sample occur, it is impossible to define a strict transition temperature, because the dynamic process leading to the glassy phase is continuous, albeit very sudden. This is indeed one of the most tricky points in the study of glassy systems: the so-called glass transition cannot actually be associated with the genuine divergence of any dynamic or thermodynamic quantity. In fact, its very definition as a reference temperature is based upon convention: in supercooled liquids, it has been agreed to fix the glass transition temperature T_g at the point where the viscosity of the sample (which is one of the macroscopic manifestations of the relaxation time) is of order $10^{13}\,P$.

On the other hand, in the case of fragile supercooled liquids [1] there is at least another important value of the temperature which is useful to describe and interpret experimental data:

that is, the temperature $T_{\rm MCT}$ where mode coupling theory (MCT) locates a purely dynamic transition [2]. Such a transition is spurious, since what is observed in real experiments and simulations is just a dynamical crossover from a diffusive regime to an Arrhenius (or super-Arrhenius) one. However, MCT describes well the dynamics of fragile liquids for $T > T_{\rm MCT}$ and, even though the MCT transition is smeared out in reality, the MCT temperature still remains a meaningful reference value marking the border between purely diffusive and activated slow dynamics [3].

The lack of a strict dynamic transition is not common to all glassy systems. It has been discovered in the past [4] that some *mean-field* models for spin-glasses display a phenomenology quite similar to the one of real structural glasses and supercooled liquids, but for a notable difference: in these mean-field systems there is a true divergence of the relaxation time, with no associated thermodynamic anomaly. This fact makes the definition of a dynamic critical temperature $T_{\rm d}$ completely unambiguous for these models. Besides, the dynamical equations which describe the behaviour of these systems above $T_{\rm d}$ coincide with those obtained by MCT [5]. Thus, for these models MCT is exact, and $T_{\rm d}$ therefore coincides with $T_{\rm MCT}$. The most deeply studied among these systems is the p-spin spherical model [6–12], henceforth indicated as pSM.

A key feature of the pSM is the possibility to explain the dynamic glass transition at $T_{\rm d}$ as the result of a purely geometric transition taking place in the energy landscape of the system [8–10]. At a given energy density, called *threshold energy* $E_{\rm th}$, there is a qualitative change in the stability properties of the landscape: below $E_{\rm th}$ and down to the ground state energy E_0 minima dominate, whereas above $E_{\rm th}$ unstable saddles are the most numerous stationary points of the Hamiltonian. It can be proved that in such a system the dynamic glass transition occurs when the equilibrium energy density becomes equal to the energy density of the threshold states at that temperature [8, 9]. In other words, in the pSM the dynamic glass transition at $T_{\rm d}$ and the geometric transition at $E_{\rm th}$ are essentially two faces of the same phenomenon. Due to this fact, the structure and properties of unstable stationary points in the pSM have been the object of a number of investigations in recent years [12–14].

Infinite-lifetime metastable states cannot exist in finite-dimensional systems, and therefore we cannot expect to find a divergence of the relaxation time with no associated thermodynamic transition in non-mean-field models. However, a strict geometric transition at a threshold energy may very well occur also in more realistic systems, such as supercooled liquids, even if its dynamical counterpart is smeared out by the finite-dimensional nature of the system (i.e. by the finite lifetime of the threshold minima). If this were true, the dynamic crossover at $T_{\rm MCT}$, which in supercooled liquids marks the onset of activated glassy dynamics, would actually be the manifestation of a more fundamental and sharply defined geometric transition occurring at a certain critical threshold energy.

This scenario has been numerically investigated very recently in [15, 16] for various Lennard-Jones (LJ) systems, and from a more speculative point of view in [17]. In particular, in [15] a well defined threshold potential energy has been located and associated to the onset of glassy dynamics in the system. Moreover, it has been directly shown in [19] that the energy landscape of LJ models and that of the pSM are indeed very similar. These studies seem therefore to confirm the idea that, even in realistic systems, the dynamic crossover observed at $T_{\rm MCT}$ is the consequence of the sharp change in the topology of the energy landscape at the threshold energy.

The approaches of [15–17] all have as a vital starting point the assumption that the time evolution of the system in the phase space is in some way influenced by the nearby saddle points of the potential energy. However, this fact has not been directly proved, and the circumstantial evidence is mainly of a numerical nature. In particular, it may be objected

that the trajectory of the system at equilibrium is never even *close* to saddles, especially above the glass transition, where it may be argued that free diffusion in the phase space implies that the energy landscape and its stationary points are completely irrelevant (for the relevance of saddles in zero-temperature dynamics: see [13]). Even in the pSM, there has been, up to now, little evidence of any *direct* connection between dynamics of the system equilibrated above $T_{\rm d}$ and saddle points of the energy above $E_{\rm th}$ (see, however, the approach of [14]). Furthermore, even assuming that the dynamic trajectory stays somewhat close to saddles, it remains to be directly demonstrated that such objects do play a role in the transition. More precisely, one should prove that the properties of these supposedly close saddles indeed display some anomaly at the dynamic transition.

The aim of this paper is therefore to analytically investigate what is the role of saddles in the equilibrium dynamics of the pSM above $T_{\rm d}$. In order to do this we introduce a tool which allows for the exact location of the closest saddle points to an equilibrium configuration at temperature T. In this way we will be able to study how the properties of these closest saddles vary with the temperature when the dynamic transition is approached, thus answering some of the questions raised above.

The Hamiltonian of the pSM is given by

$$H = \sum_{i_1<\cdots<i_p}^{N} J_{i_1\cdots i_p}\tau_{i_1}\cdots\tau_{i_p} = \frac{1}{p!}\sum_{i_1\cdots i_p}^{N} J_{i_1\cdots i_p}\tau_{i_1}\cdots\tau_{i_p} + \mathrm{O}(1/N) \tag{1}$$

where the spins satisfy the spherical constraint $\sum_i \tau_i^2 = N$. The quenched couplings $J_{i_1\cdots i_p}$ are Gaussian-distributed random variables with variance $J^2 = p!/2N^{p-1}$. By means of the Lagrange method, we can find the stationary points of the Hamiltonian on the sphere and therefore write the equations satisfied by the saddle points of H with energy density E:

$$\frac{1}{p!}\sum_{i_2\cdots i_p}^{N} J_{k,i_2\cdots i_p}\tau_{i_2}\cdots\tau_{i_p} - E\tau_k = 0 \qquad k = 1,\ldots,N. \tag{2}$$

In general, the number $\mathcal{N}(E)$ of solutions of equations (2) with energy density E is exponentially large in the size of the system N. Thus, the quantity which is normally computed is the *complexity* (or configurational entropy), defined as the logarithmic density of this number, $\Sigma(E) = \frac{1}{N}\log\mathcal{N}(E)$. The nature of the saddle points of H is, in principle, not only specified by their energy density E, but also by their instability index K, that is the number of negative eigenvalues of the Hessian matrix. However, previous studies of the pSM have shown that there is a well defined relation between energy density and index $K(E)$, and that *at any fixed energy level* E only stationary points with index $K(E)$ dominate the energy landscape in the thermodynamic limit [9,12,18]. Therefore, by fixing the energy density of a saddle point to E, we are automatically fixing its index to $K(E)$. A crucial feature of the pSM is that the typical saddle index is extensive, $K = \mathrm{O}(N)$, as long as the energy density is above a value called *threshold*, $E_{\rm th}$, while $K = 0$ for $E \leqslant E_{\rm th}$. This means that minima dominate over saddles below the threshold, while saddles of index $K(E) > 0$ are the most numerous stationary points for $E > E_{\rm th}$. In this sense, we can say that at $E_{\rm th}$ a geometric transition takes place. More precisely, if we introduce the index density $k = K/N$, we have

$$k(E) = \frac{p}{\pi(p-1)}\left[\arctan\left(-\frac{\sqrt{E_{\rm th}^2 - E^2}}{E}\right) + \frac{E}{4}\sqrt{E_{\rm th}^2 - E^2}\right] \qquad E \geqslant E_{\rm th}$$
$$k(E) = 0 \qquad E \leqslant E_{\rm th}. \tag{3}$$

Note that $k(E)$ is a monotonically increasing function of the energy E. Remarkably, when the equilibrium energy density of the system becomes equal to the internal energy density of the threshold minima the system undergoes a dynamic glass transition, which we will indicate with $T_{\rm d}$. To better specify this statement, we have to distinguish between the *bare* energy E of a minimum, and its internal energy $U(T)$: that is, the energy of a system equilibrated in that minimum at temperature T[4]. Of course, the quantity $U(T)$ is equal to the bare energy E plus a vibrational contribution due to thermal fluctuations. At the dynamic glass transition $T_{\rm d}$ we have that $U_{\rm eq}(T_{\rm d}) = U_{\rm th}(T_{\rm d})$, where $U_{\rm eq}(T)$ is the global equilibrium energy density of the system, and $U_{\rm th}(T_{\rm d}) = E_{\rm th}+$ vibrations.

Our aim in this paper is to analyse the structure of the saddle points around an equilibrium configuration thermalized at temperature $T > T_{\rm d}$. We therefore need a notion of distance to give a meaning to this statement. Given two configurations σ and τ we define a co-distance, or *overlap*, $q_{\sigma\tau}$, as

$$q_{\sigma\tau} = \frac{1}{N}\sum_{i}^{N}\sigma_i\tau_i.$$

Similar configurations have $q \sim 1$, while different ones have $q \sim 0$. Our strategy will be to fix a reference equilibrium configuration σ and compute the complexity of the saddles points τ close to it as a function of their overlap $q_{\sigma\tau}$. The value of the overlap where this quantity goes to zero will give the distance of the closest stationary points to σ. Indeed, for larger overlaps, i.e. smaller distances, a negative complexity indicates a vanishing probability of finding a stationary point.

In order to do this we have to calculate how many saddles τ, with a given energy E, happen to have an overlap q with a reference equilibrium configuration σ. Clearly, this number formally depends on σ itself and on the disorder J. However, as always done in similar calculations [11, 20, 21], we can assume that in the thermodynamic limit $N \to \infty$ this quantity is self-averaging with respect to the distribution of σ and J, and therefore we can average it over the Gaussian distribution of the disorder (indicated with a bar) and over the equilibrium distribution of σ at temperature T. In this way we can define the saddle complexity as

$$\Sigma_s(q, E, \beta) \equiv \frac{1}{N}\overline{\int \frac{{\rm D}\sigma}{Z(\beta)}\, {\rm e}^{-\beta H(\sigma)}} \times \overline{\log \int {\rm D}\tau \ \prod_k \delta\left(\frac{1}{p!}J_{k,i_2\cdots i_p}\tau_{i_2}\cdots\tau_{i_p} - E\tau_i\right)\left|\det\left(\frac{1}{p!}J_{k,l,i_3\cdots i_p}\tau_{i_3}\cdots\tau_{i_p} - E\delta_{kl}\right)\right| \ \delta(q - q_{\sigma\tau})} \tag{4}$$

with

$$Z(\beta) = \int {\rm d}\sigma\, {\rm e}^{-\beta H(\sigma)}$$

and where integration is carried out over spherical configurations only. To understand equation (4) it is convenient to read it from right to left: first, under the τ integral, we calculate using the standard method of [22] the number of solutions of equations (2), putting an extra constraint on the overlap they must have with σ. Second, we take the logarithm of this quantity, and we average it over the equilibrium distribution of σ. Finally, we average everything over the disorder J. As we can see, Σ_s depends on the temperature $T = 1/\beta$ at which the reference configuration σ is equilibrated, on the energy E of the saddles τ we are counting, and finally on

[4] In other terms, if m_i is the local magnetization of the system equilibrated in the minimum, and $q = 1/N\sum_i m_i^2$ is the *self-overlap* (i.e. the magnetization norm, which is related to thermal fluctuations), we can define a bare magnetization as $\hat{m}_i = \frac{m_i}{\sqrt{q}}$. The bare energy density is then given by $E = \frac{1}{N}H(\hat{m}_i)$.

the overlap q between σ and τ. We stress that, by construction, σ is an independent equilibrium configuration, irrespective of the energy and the distance of the saddle τ.

In order to perform the averages in (4) it is convenient to use the replica method, writing

$$\begin{aligned} Z^{-1} &= \lim_{n\to 0} Z^{n-1} \\ \langle \log(\cdot)\rangle &= \lim_{m\to 0} \frac{1}{m}\log\langle(\cdot)^m\rangle. \end{aligned} \tag{5}$$

In this way we have

$$\Sigma_s(q,E,\beta) = \lim_{n,m\to 0}\frac{1}{Nm}\log\overline{\int \mathrm{D}\sigma_a\,\mathrm{D}\tau_\alpha\,\mathrm{e}^{-\beta\sum_a H(\sigma_a)}} \times\overline{\prod_{k\alpha}\delta\left(\frac{1}{p!}J_{k,i_2\cdots i_p}\tau^\alpha_{i_2}\cdots\tau^\alpha_{i_p}-E\tau^\alpha_i\right)\left|\det\left(\frac{1}{p!}J_{k,l,i_3\cdots i_p}\tau^\alpha_{i_3}\cdots\tau^\alpha_{i_p}-E\delta_{kl}\right)\right|\,\delta(q-q_{\sigma_1\tau_\alpha})} \tag{6}$$

with $a = 1,\ldots,n$ and $\alpha = 1,\ldots,m$. The explicit calculation of Σ_s from equation (6) can be performed by using the standard tools of the replica method: variational parameters are introduced and the integrals are evaluated exactly in the limit $N\to\infty$ by means of the steepest descent method. Of course, this is possible thanks to the mean-field nature of the model. Here, we will skip most of the details and just state the final result. The interested reader may refer to [11], where a technically similar calculation is performed. The full expression for the saddle complexity is

$$\begin{aligned} \Sigma_s(q,E,\beta) = {}& \frac{1}{2}+\frac{pE^2}{2(p-1)}+\frac{1}{2}\log\left(\frac{p-1}{2p}\right)+\frac{1}{4p}(x_1-x_0r^{p-1})+\frac{1}{2}\beta q^{p-1}w+Ey_1 \\ & +\frac{p-1}{4p}\left(y_1^2-y_0^2r^{p-2}\right)+\frac{1}{2}\log\Omega_1+\frac{\Omega_1}{2\Omega_2}+\frac{1}{2}\left(\frac{r-q^2}{1-r}\right) \end{aligned} \tag{7}$$

with

$$\begin{aligned} \Omega_1 &= (x_1-x_0)(1-r)+(y_1-y_2)^2 \\ \Omega_2 &= (x_0+w^2)(1-r)+(y_1-y_0)[2(y_0-qw)-(r-q^2)(y_1-y_0)/(1-r)]. \end{aligned} \tag{8}$$

As customary in the context of the replica method, the set of variational parameters $\boldsymbol{x} = (x_0, x_1, y_0, y_1, w, r)$ is fixed be means of the steepest descent equations $\partial\Sigma_s/\partial\boldsymbol{x} = 0$, which we have solved numerically. We remark that the expression above for Σ_s is only valid in the regime $E \geqslant E_{\mathrm{th}}$. In showing the results we will assume $p = 3$.

First of all, we are interested in studying the behaviour of $\Sigma_s(q, E, T)$ as a function of q, at fixed E and T. In this way we can define an overlap $q_0(E,T)$ where Σ_s goes to zero: this overlap gives the distance of the closest saddle with energy E to an equilibrium configuration at temperature T. In figure 1 we plot Σ_s as a function of the overlap q, for $T = T_{\mathrm{d}}$ and $E = E_{\mathrm{th}}$. At this temperature many properties of equilibrium landscape are known and an interpretation of the results is therefore much simpler. At T_{d} the system equilibrates inside a threshold state with bare energy density E_{th} and self-overlap (i.e. largeness) q_{th} [9]. In naive terms we can then imagine that our typical equilibrium configuration σ lies in a well whose largeness is given by q_{th} and whose bottom is at energy density E_{th}. In this case it is evident that the closest stationary point to σ is precisely the bottom of the well. The point where the complexity goes to zero must therefore give the overlap between the centre of the threshold minimum τ and one of its typical equilibrium configurations σ. This overlap can be easily computed by noticing that $q_{\sigma\tau} = \frac{1}{N}\sum_i\langle\sigma_i\tau_i\rangle = \frac{1}{N}\sum_i m_i\tau_i$, where the thermal average is restricted to a threshold state, and m_i indicates the local magnetization of that state. In the pSM this local

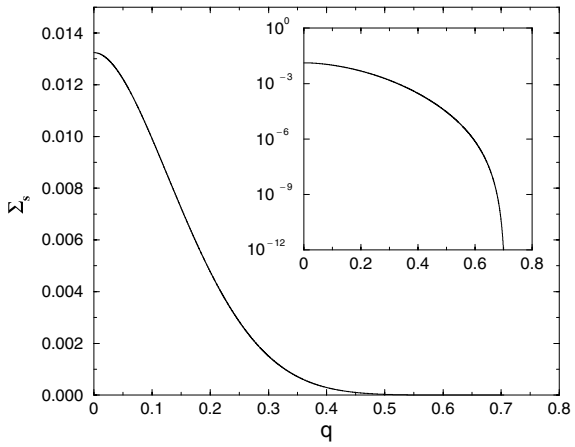

Figure 1. Saddle complexity as a function of the overlap q, for $E = E_{\rm th}$ and $T = T_{\rm d}$. Inset: same curve in linear–log scale. The complexity goes to zero at $q_0 = \sqrt{q_{\rm th}} = 0.71$ for $p = 3$.

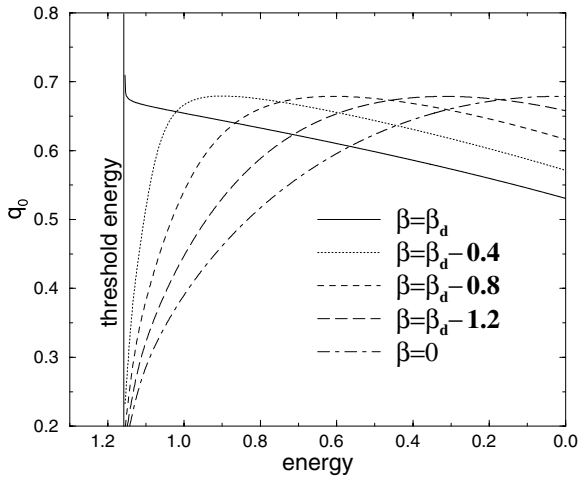

Figure 2. The overlap q_0 where the saddle complexity goes to zero as a function of the energy E of the saddles, at five different values of the temperature above $T_{\rm d}$. The full vertical line marks the value of the threshold energy $E_{\rm th}$.

magnetization can be expressed directly in terms of the well minimum as $m_i = \sqrt{q_{\rm th}}\tau_i$ [9] (see also footnote 4) and we immediately get $q_{\sigma\tau} = \sqrt{q_{\rm th}}$. Consistently with this result we find that

$$q_0(E_{\rm th}, T_{\rm d}) = \sqrt{q_{\rm th}}. \tag{9}$$

This result can be appreciated in figure 1.

Moreover, a careful analysis of Σ_s for $q \sim q_0$ shows that

$$\Sigma_s(q, E_{\rm th}, T_{\rm d}) \sim (q - q_0)^5 \qquad q \sim q_0. \tag{10}$$

Note that the exponent is the same as found in [11] for the approach to zero of the constrained complexity of threshold Thouless–Anderson–Palmer (TAP) solutions. This is a consistency check for the present calculation.

In the light of our original aim to find the closest saddles, it is interesting to plot the value of q_0 as a function of the energy density E of the saddles we are counting. We expect this

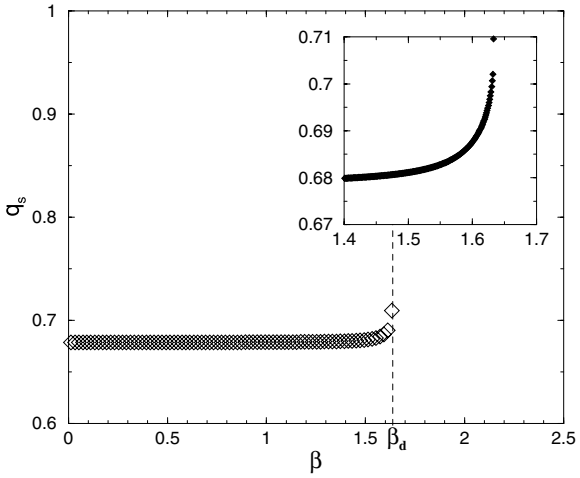

Figure 3. The overlap q_s of the closest saddles to an equilibrium configuration at temperature T, as a function of $\beta = 1/T$. Inset: enlargement of the same plot for $\beta \sim \beta_{\rm d}$.

curve to have a maximum at a value $E_s(T)$ corresponding to the energy of the closest, and thus the most relevant, saddles. Accordingly, in figure 2 we plot $q_0(E)$ as a function of E for different values of the temperature. In particular, the full curve represents $q_0(E)$ for $T = T_{\rm d}$: as we see, this is a steadily decreasing curve having its maximum at the threshold energy. This means that, as previously said, the closest stationary point to an equilibrium configuration at the glass transition is a threshold minimum, $E_s(T_{\rm d}) = E_{\rm th}$. However, if we now increase the temperature T of the equilibrium configuration, we expect the energy of the closest saddle to increase as well, together with its instability index. In other words, the higher the temperature of σ, the higher will be the energy, and thus the degree of instability, of the closest possible saddle (we remind that $K(E)$ is a monotonic increasing function of E, with $K(E_{\rm th}) = 0$). This hypothesis is confirmed in figure 2: the maximum of these curves moves to the right as the temperature is increased, disclosing a well defined relation $E_s(T)$, which we will analyse carefully later. For the moment, let us note that a further consistency check of our calculation is that for $T = \infty$ we find $E_s = 0$ (see figure 2). At very high temperatures the equilibrium configuration σ is just a random configuration of the system, therefore the closest stationary points to it will be the most numerous ones in absolute terms. In the pSM it can be proved that the most numerous saddles have $E = 0$ and $K = N/2$ [9, 18].

It is interesting to study the behaviour of the overlap q_s where the curves $q_0(E)$ have their maximum. This overlap is a measure of the closeness of an equilibrium configuration σ to its nearest saddle point τ. What is surprising about figure 2 is that by varying the temperature the value of q_s remains almost constant. To better investigate this point we plot in figure 3 q_s as a function of β. We can see that q_s is practically always constant, except for $\beta \sim \beta_{\rm d}$, where it sharply jumps to $\sqrt{q_{\rm th}}$. This fact means that *the distance between an equilibrium configuration and its closest saddles is almost independent of the temperature*. This value of the overlap is $q_s \sim 0.68$, which is indeed quite high, being comparable to the overlap between equilibrium configurations and the bottom of the minima below the glass transition (see, for example, figure 1). This result answers one of the main questions raised in the introduction: above the dynamic glass transition, the equilibrium trajectory indeed always stays very close to *unstable* stationary points of the Hamiltonian, exactly as below $T_{\rm d}$, it stays close to *stable* minima. On the other hand, it is clear from figure 2 that for $T > T_{\rm d}$ the closest minima (i.e.

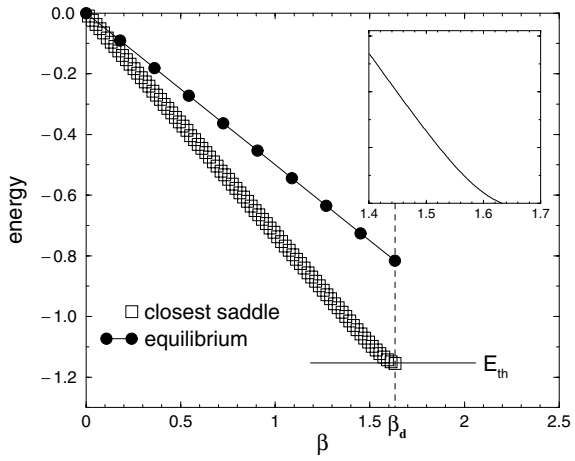

Figure 4. The bare energy E_s of the closest saddles compared with the equilibrium energy $U_{\rm eq}$. Inset: enlargement of the same plot for $\beta \sim \beta_{\rm d}$. The slope of the curve changes in this regime.

the ones with $E = E_{\rm th}$) are very far from the dynamic trajectory. In this sense, it is justifiable to say that the equilibrium dynamics of the system above the glass transition may be described as an evolution among the neighbourhoods of saddle points, rather than among basins of the minima [17].

The distance of the closest saddle does not change with T but, as we have seen, the energy density E_s does. In figure 4 we plot this energy as a function of the inverse temperature and compare it with the equilibrium energy density of the system above the glass transition, i.e. $U_{\rm eq}(T) = -\beta/2$ [6]. First of all we note that the energy of the saddle is always smaller than the energy of the equilibrium configuration, despite the two objects being so close in the phase space. This fact has been already noted in the context of a numerical study of a LJ system in [16]. It is tempting to interpret the difference $U_{\rm eq}(T) - E_s(T)$ as a pseudo-vibrational contribution of saddles, due to the fact that, even though $K > 0$, the largest part of the Hessian eigenvalues is positive, as long as $E_{\rm th} < E < 0$. Unstable saddles are not trapping objects, of course, but they may have a substantially long life-time provided that K is small enough. This phenomenon is at the basis of the pseudo-vibrational contribution of saddles.

This last hypothesis is supported by another interesting result we find, that is

$$E_s(T) \to E_{\rm th} \qquad T \to T_{\rm d}. \tag{11}$$

Therefore, the energy density of the closest saddles to an equilibrium configuration goes to the threshold energy density at the dynamic glass transition. Clearly, at $T_{\rm d}$ the difference between $E_s(T_{\rm d})$ and $U_{\rm eq}(T_{\rm d})$ is given by the vibrational contribution of thermal fluctuations inside threshold minima, which is of order $k_{\rm B}T$. When $T > T_{\rm d}$ we see that the two curves continuously approach one another, as the saddle instability index K increases with the energy. These results seem thus to suggest that, even though the system is *not* confined into any given saddle point, the disproportion between trapping and un-trapping directions, that is the fact that $K < N/2$, is sufficient to produce a vibrational contribution that we may broadly interpret as thermal fluctuations around saddle points.

Further support for this idea comes from the comparison of our results with the approach developed in [14], where a dynamic description of the pSM based on the concept of quasi-states is introduced. More specifically, for temperatures slightly above the dynamic transition $T_{\rm d}$, quasi-states are related to critical points of the TAP free energy (for a more detailed definition

see [14]) and it can be shown that their global contribution gives rise to the paramagnetic free energy of the system. Interestingly enough, we found that the bare energy density of these quasi-states (defined as in footnote 4) is to a good degree of accuracy (within 1%) equal to the energy density E_s of the closest saddles. This indicates that the quasi-states introduced in [14] may be interpreted as our closest saddles plus the pseudo-vibrational contribution mentioned before. This interpretation confirms the idea, outlined in [14] and made more explicit in the present paper, that above T_d the paramagnetic state is made up of disjoint quasi-states around saddles. Equilibrium dynamics can thus be thought of as evolution from one quasi-state (i.e. one saddle and its own neighbourhood) to another one. A difference between the present approach and the one of [14] is that the pseudo-states of [14] can be defined only very close to T_d, while, as we have seen, closest saddles exist at any temperature, although, of course, we do not expect them to have any relevance for $T \gg T_d$.

Given the relation $k(E)$ between index and energy of the typical saddles, we can introduce a temperature-dependent index by using the energy density $E_s(T)$ of the closest saddles, namely $k(T) = k(E_s(T))$. Clearly, this index vanishes at the dynamic transition temperature T_d, which is just another way of describing the geometric transition occurring at the threshold energy E_{th}. Close to T_d we find that E_s is linear in β and this, together with the analytic form of $k(E)$, implies that

$$k(T) \propto (T - T_d)^{3/2} \qquad T \sim T_d. \tag{12}$$

Summarizing, in the context of the pSM we have calculated the complexity of the saddle points at fixed overlap with a reference equilibrium configuration, above the dynamic glass transition. In this way we were able to identify what are the energy and distance of the closest saddles at any given temperature. We found that the distance between equilibrium configuration and closest saddle is almost independent of the temperature and is very small. Moreover, the energy of the closest saddles intersects the threshold energy at the dynamic glass transition. Finally, we interpreted the difference between equilibrium energy and energy of the closest saddle as a pseudo-vibrational contribution due to the fact that a number of trapping directions larger than the number of non-trapping ones may give rise to thermal fluctuations around unstable saddle points. The present study supports the idea that dynamics in glassy systems for $T > T_d$ can be described in terms of evolution in the phase space among the neighbourhoods of unstable saddles and strengthens the hypothesis that the glass transition, even in finite-dimensional systems, is just the manifestation of the topological transition between saddles and minima dominated regions of the phase space.

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